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Density results for Gabor systems associated with periodic subsets of the real line

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Abstract

The well-known density theorem for one-dimensional Gabor systems of the form $\{e^{2\pi imbx}\,g(x-na)\}_{m,n\in\mathbb{Z}}$, where $g\in L^2(\mathbb{R})$, states that a necessary and sufficient condition for the existence of such a system whose linear span is dense in $L^2(\mathbb{R})$, or which forms a frame for $L^2(\mathbb{R})$, is that the density condition $a\,b\le 1$ is satisfied. The main goal of this paper is to study the analogous problem for Gabor systems for which the window function g vanishes outside a periodic set $S\subset\mathbb{R}$ which is $a\,\mathbb{Z}$-shift invariant. We obtain measure-theoretic conditions that are necessary and sufficient for the existence of a window g such that the linear span of the corresponding Gabor system is dense in $L^2(S)$. Moreover, we show that if this density condition holds, there exists, in fact, a measurable set $E\subset\mathbb{R}$ with the property that the Gabor system associated with the same parameters a,b and the window $g=\chi_E$, forms a tight frame for $L^2(S)$.

Keywords: Subspace Gabor frames; Riesz bases; Zak transform; Density of Gabor systems

1. Introduction

The theory of frames was first introduced in 1952 in a paper by Duffin and Schaeffer ([6]; see also [18]) dealing with nonharmonic Fourier series. It came back into the limelight in recent years with the apparition of a large number of papers dealing with specific applications of frames, mostly to wavelets and Gabor systems. Let us briefly recall some important definitions and results of the theory of frames. If $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, with

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inner product $\langle ., . \rangle$, and $\mathcal{N}$ is a countable index set, we say that a collection $X = \{x_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ is a *frame* for its closed linear span $\mathcal{M}$ if there exist constants $A, B > 0$, called the *frame bounds*, such that

$$A\|x\|^2 \le \sum_{n\in\mathcal{N}} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad x \in \mathcal{M}. \tag{1.1}$$

A frame X is said to be *tight* (resp. *a Parseval tight frame*) if $A = B$ (resp. $A = B = 1$) in (1.1). We call the collection X *Bessel*, with constant B, if the second inequality in (1.1) holds for all $x \in \mathcal{M}$. The collection $X = \{x_n\}_{n\in\mathcal{N}}$ is called a *Riesz family* or *Riesz sequence* with constants C, D, if the inequalities

$$C\sum_{n\in\mathcal{N}} |a_n|^2 \le \left\|\sum_{n\in\mathcal{N}} a_n x_n\right\|^2 \le D\sum_{n\in\mathcal{N}} |a_n|^2,$$

hold for all (finitely supported) sequences $\{a_n\}$ of complex numbers. If the linear span of a Riesz family X is dense in $\mathcal{H}$, we say that X forms a *Riesz basis*. If we can choose $C = D = 1$, then X is an *orthonormal family* and an *orthonormal basis* if its closed linear span is $\mathcal{H}$. Let $\ell^2(\mathcal{N})$ denote the space of complex-valued square-summable sequences indexed by $\mathcal{N}$. If X is a Bessel collection, the *analysis operator* or *frame transform* associated with X, $T_X : \mathcal{M} \to \ell^2(\mathcal{N})$, is defined by

$$T_X(x) = \{\langle x, x_n\rangle\}_{n\in\mathcal{N}}, \quad x \in \mathcal{M}, \tag{1.2}$$

while its adjoint, the *synthesis operator*, $T_X^* : \ell^2(\mathcal{N}) \to \mathcal{M}$, is given by

$$T_X^*(\{c_n\}_{n\in\mathcal{N}}) = \sum_{n\in\mathcal{N}} c_n\, x_n, \quad \{c_n\}_{n\in\mathcal{N}} \in \ell^2(\mathcal{N}). \tag{1.3}$$

The *frame operator* $\mathcal{S}$ is defined by $\mathcal{S} = T_X^* T_X : \mathcal{M} \to \mathcal{M}$. More explicitly

$$\mathcal{S}x = \sum_{n\in\mathcal{N}} \langle x, x_n\rangle\, x_n, \quad x \in \mathcal{M}. \tag{1.4}$$

If X is a frame for $\mathcal{M}$, then $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is a bounded, positive and invertible operator with a bounded inverse. The collection $\{\mathcal{S}^{-1}x_n\}_{n\in\mathcal{N}}$ is called the *standard dual frame* of the frame X and we have the reconstruction formula

$$x = \sum_{n\in\mathcal{N}} \langle x, x_n\rangle\, \mathcal{S}^{-1}x_n = \sum_{n\in\mathcal{N}} \langle x, \mathcal{S}^{-1}x_n\rangle\, x_n, \quad x \in \mathcal{M}.$$

In the following, we will let $\mathcal{H} = L^2(\mathbb{R})$ and consider expansions in terms of one-dimensional Gabor (also called Weyl–Heisenberg) systems of the form $\mathbf{G} = \{e^{2\pi imbx}\, g(x - na)\}_{m,n\in\mathbb{Z}}$, where $a, b > 0$ are two real parameters and g is a function in $L^2(\mathbb{R})$ called the *window* function. Such systems have been studied quite extensively, mostly when the expansions are considered on the whole space $L^2(\mathbb{R})$ (see [13,7,8,15,16] and the references therein), but also in the context of subspaces (as in [2,10–12]).

In the one-dimensional case, the well-known density theorem for Gabor systems states that a necessary and sufficient condition for the existence of a Gabor system $\mathbf{G}$ as above whose linear span is dense in $\mathcal{H} = L^2(\mathbb{R})$ is that $a\,b \le 1$. Moreover, if this last condition holds, there exists a function $g \in L^2(\mathbb{R})$ such that the associated system $\mathbf{G}$ forms a tight frame for $L^2(\mathbb{R})$. In fact, it is not difficult to show that $g = \chi_{[0,a)}$ will do the trick. The necessary (and harder) part of this result was first obtained by Daubechies [5] in the rational case (i.e. $a\,b \in \mathbb{Q}$) and is generally attributed to Baggett [1] and Rieffel [17] in the irrational one. (See [16] for more information on the history of this result.)

The main goal of this paper is to study related problems for subspaces of $L^2(\mathbb{R})$ of the form $L^2(S) = \{f \in L^2(\mathbb{R}),\ f = 0 \,\text{a.e. on}\, \mathbb{R} \setminus S\}$, where S is a measurable subset of $\mathbb{R}$ which is $a\,\mathbb{Z}$-shift invariant, i.e. S has the property that it is invariant under the transformation $x \mapsto x + a$. If g itself vanishes a.e. outside of S, it is clear that the closed linear space generated by the corresponding system $\mathbf{G}$ will be a subspace of $L^2(S)$. One can then ask for conditions on S depending on a, b for the existence of a system $\mathbf{G}$ whose linear span is dense in $L^2(S)$. If this condition holds, one can then ask if there exist such collections $\mathbf{G}$ forming a (tight) frame, Riesz basis, etc. for $L^2(S)$. This framework can model a situation where a signal is known to appear periodically but intermittently and one would try to perform a Gabor analysis of the signal in the most efficient way possible while still preserving all the features of the observed data. One could think of the signal as existing for all time t and do the analysis in the usual way but clearly, if the signal is only emitted for very short periods of time, this might not be the optimal way to proceed. Since the correct density condition is $a\,b \leq 1$ in the case where $S = \mathbb{R}$, one would assume that if S is "smaller" than $\mathbb{R}$, a corresponding smaller density condition might result. One might guess that the correct density condition should be that $b\,|S \cap [0, a)| \leq 1$, where $|\cdot|$ denotes the Lebesgue measure. In fact, that condition was proved to be necessary in [11]. As we will show, it turns out to be the right density condition in the irrational case, but not in the rational one. More precisely, we will prove that, if $a\,b = \frac{p}{q}$, where p and q are two positive integers with $\gcd(p, q) = 1$, the correct density condition is that $\sum_{k=0}^{p-1} \chi_S(\cdot + \frac{k}{b}) \leq q \quad \text{a.e. on } \mathbb{R}$. One of our main results, is that, in both cases, if the appropriate density condition is satisfied, we can construct a window g of the form $g = \chi_E$, where E is a measurable subset of $\mathbb{R}$ with finite measure, such that the corresponding system $\mathbf{G}$ actually forms a tight frame for $L^2(S)$. In fact, we will show that the possibility of constructing a Gabor subspace frame of this form for $L^2(S)$ is equivalent to being able to solve a certain tiling problem related to the set S and the density condition is exactly what is needed for the tiling problem to have a solution. We note that the idea of using a window which is the characteristic function of a measurable set was also used by Han and Wang [14] to show the existence of Gabor frames (where the parameters a, b are replaced by invertible matrices) in higher dimensions for the space $L^2(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we consider the rational case. Given a measurable subset S of the real line, invariant by $a\,\mathbb{Z}$-translations, and a window $g \in L^2(S)$, we provide a necessary and sufficient condition for the linear span of the system $\{e^{2\pi i m b x}\, g(x - na)\}_{m,n\in\mathbb{Z}}$ to be dense in $L^2(S)$ under the assumption that the product $a\,b$ is a rational number p/q where $\gcd(p, q) = 1$ (Theorem 2.7). This condition involves the rank of a $q \times p$ matrix-valued function $\mathcal{G}$ built using the Zak transform of g and implies the density condition for the rational case mentioned earlier. Using an iterative construction using finitely many steps (in fact, q steps), we show that if this density condition is satisfied, then there exists a measurable set $E \subset \mathbb{R}$ with $|E| < \infty$, such that the Gabor system associated with the window $g = \chi_E$ actually forms a tight frame for $L^2(S)$ (Theorem 2.12). In Section 3, we give a proof of the fact that the condition $b\,|S \cap [0, a)| \leq 1$ is necessary in order for a Gabor system as above to form a frame for $L^2(S)$, whether $a\,b$ is rational or not, and that, if such a frame exists, it will form a Riesz basis if and only if $b\,|S \cap [0, a)| = 1$ (Theorem 3.3). Finally, we show in Section 4, for the irrational case, that if the density condition $b\,|S \cap [0, a)| \leq 1$ holds, one can again construct a window function of the form $g = \chi_E$ such that the associated system forms a tight frame for

$L^2(S)$ (Theorem 4.2). The construction of E is done using a similar iterative procedure as for the rational case, but requiring now an infinite number of steps.

2. The rational case

In this section, we will consider Gabor systems of the form

$$\mathbf{G} = \{e^{2\pi i mbx}\, g(x - na)\}_{m,n\in\mathbb{Z}},$$

where $a\,b \in \mathbb{Q}$ and g is a window vanishing a.e. outside of a set S which is $a\,\mathbb{Z}$-shift invariant. The Zak transform will be one of the main tools used in this section, which is not unusual when dealing with Gabor systems in the rational case (see [5,13]). We first introduce some notations and definitions.

Let E be a measurable set in $\mathbb{R}$ with nonzero Lebesgue measure (which will be denoted by $|E|$). We identify $L^2(E)$ with $\{f \in L^2(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R}\setminus E\}$. For $x, \omega \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$, we denote by T_x and E_ω, the translation and modulation operators defined respectively by

$$(T_x g)(t) = g(t - x) \quad \text{and} \quad (E_\omega g)(t) = e^{2\pi i\omega t}\, g(t), \quad t \in \mathbb{R}.$$

For a fixed $\alpha > 0$, we define the Zak transform $\mathcal{Z}_\alpha : L^2(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}^2)$ to be the mapping

$$(\mathcal{Z}_\alpha f)(t, v) = \sum_{k\in\mathbb{Z}} f(t + k\alpha)e^{2\pi ikv}, \quad f \in L^2(\mathbb{R}),$$

defined for a.e. $(t, v) \in \mathbb{R}^2$. It is easy to check that

$$(\mathcal{Z}_\alpha f)(t + k\,\alpha, v + l) = e^{-2\pi ikv}\,(\mathcal{Z}_\alpha f)(t, v) \tag{2.1}$$

for $k, l \in \mathbb{Z}$ and a.e. $(t, v) \in \mathbb{R}^2$. The interested reader can consult [13, Chapter 8] for further information on the Zak transform. The following lemma, although quite elementary, will play an important role in our analysis of the rational case. With a slight abuse of language, we will call the operator that maps a function $f \in L^2(\mathbb{R})$ to the restriction of its Zak transform $\mathcal{Z}_\alpha f$ to a subset of $\mathbb{R}^2$, the restriction of the Zak transform $\mathcal{Z}_\alpha$ to that subset.

Lemma 2.1. *Let S be a measurable subset of $\mathbb{R}$ which is $\alpha\mathbb{Z}$-shift invariant and define $S_0 = S \cap [0, \alpha)$. Then,*

- *The restriction of $\mathcal{Z}_\alpha$ to the set $[0, \alpha) \times [0, 1)$ is a unitary operator from $L^2(\mathbb{R})$ onto $L^2([0, \alpha) \times [0, 1))$.*
- *The image of $L^2(S)$ under the restriction of $\mathcal{Z}_\alpha$ to the set $[0, \alpha) \times [0, 1)$ is the subspace $L^2(S_0 \times [0, 1))$.*

Proof. The first statement is a well-known property of the Zak transform. To prove the second one, note first that, from the definition of $\mathcal{Z}_\alpha$, if $f \in L^2(S)$, then $\mathcal{Z}_\alpha f = 0$ a.e. on $(\mathbb{R}\setminus S) \times \mathbb{R}$. Hence, when we restrict the Zak transform to $[0, \alpha) \times [0, 1)$, we deduce that $\mathcal{Z}_\alpha f \in L^2(S_0 \times [0, 1))$. Conversely, given an arbitrary function $F(t, v) \in L^2(S_0 \times [0, 1))$, we have, for any $k \in \mathbb{Z}$,

$$(\mathcal{Z}_\alpha^{-1} F)(t + k\alpha) = \int_0^1 F(t, v)e^{-2\pi ikv}dv = 0, \quad \text{for a.e. } t \in [0, 1) \setminus S_0,$$

which shows that $\mathcal{Z}_\alpha^{-1} F \in L^2(S)$. The mapping $\mathcal{Z}_\alpha : L^2(S) \to L^2(S_0 \times [0, 1))$ is thus surjective which proves our claim. $\square$

Before stating the main results of this section, we will need the following definition and some preliminary lemmas.

Definition 2.2. If $j \in \mathbb{Z}$, we denote by τ_j the translation operator acting on the finite group $\mathbb{Z}_p$ identified with the set $\{0, 1, \ldots, p-1\}$ and defined by

$$\tau_j(k) = k - j \bmod (p), \quad k = 0, \ldots, p-1.$$

If $A \subset \{0, 1, \ldots, p-1\}$, we let $\tau_j(A) = \{\tau_j(k),\ k \in A\}$.

The following lemma is well-known.

Lemma 2.3. *Let $p_1, p_2 \in \mathbb{N}$ satisfy $\gcd(p_1, p_2) = 1$. Then, to every $j \in \mathbb{Z}$, there correspond a unique $k \in \mathbb{Z}$ and a unique $r \in \{0, 1, \ldots, p_1 - 1\}$ such that*

$$j = k\, p_1 + r\, p_2. \tag{2.2}$$

Lemma 2.4. *Let $a, b > 0$ satisfy $a\, b = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$ and let S be a measurable subset of $\mathbb{R}$ with nonzero measure, and with S being a $\mathbb{Z}$-shift invariant. Define the function*

$$h(t) := \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right), \quad t \in \mathbb{R}$$

and, given $t \in \mathbb{R}$, define the set

$$K_S(t) = \left\{k \in \{0, 1, \ldots, p-1\},\ \chi_S\left(t + \frac{k}{b}\right) = 1\right\}.$$

Then, the function h is $\frac{1}{bq}$-periodic and, furthermore, if $j \in \mathbb{Z}$, we have the identity

$$K_S\left(t + \frac{j}{bq}\right) = \tau_{k_0}(K_S(t)),$$

where k_0 is the unique integer satisfying $\frac{j}{q} = k_0 + \frac{rp}{q}$ with $r \in \{0, 1, \ldots, q-1\}$.

Proof. Letting $S_0 = S \cap [0, a)$, we have $\chi_S = \sum_{n\in\mathbb{N}} \chi_{S_0}(\cdot + n\, a)$. Thus, using Lemma 2.3 with $p_1 = p$ and $p_2 = q$, we have

$$\begin{aligned} h &= \sum_{k=0}^{p-1} \sum_{n\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{k}{b} + \frac{np}{bq}\right) = \sum_{k=0}^{p-1} \sum_{n\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{1}{bq}(kq + np)\right) \\ &= \sum_{l\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{l}{bq}\right) \end{aligned} \tag{2.3}$$

and this last expression is clearly $\frac{1}{bq}$-periodic, which proves the first part of the claim. Next, note that, for a.e. $t \in \mathbb{R}$, the mapping $k \mapsto \chi_S(t + \frac{k}{b})$ is p-periodic, since

$$\chi_S\left(t + \frac{k+p}{b}\right) = \chi_S\left(t + \frac{k}{b} + \frac{p}{bq}q\right) = \chi_S\left(t + \frac{k}{b} + aq\right) = \chi_S\left(t + \frac{k}{b}\right).$$

If $\frac{j}{q} = k_0 + \frac{rp}{q}$ as above, we have

$$\chi_S\left(t + \frac{j}{bq} + \frac{k}{b}\right) = \chi_S\left(t + \frac{k_0}{b} + \frac{rp}{bq} + \frac{k}{b}\right) = \chi_S\left(t + \frac{k + k_0}{b}\right).$$

Thus, $k \in K_S(t + \frac{j}{bq})$ if and only if $\chi_S(t + \frac{k+k_0}{b}) = 1$, which, using the periodicity property just mentioned, is equivalent to the fact that $k \in \tau_{k_0}(K_S(t))$. $\square$

In what follows, our analysis in the case where the product ab is rational, will depend in an essential way on properties of a matrix-valued function associated with the window function g and defined using the Zak transform. We denote by $\mathcal{M}_{q,p}$ the space of matrices with complex entries of size $q \times p$. A function taking values in $\mathcal{M}_{q,p}$ is said to be measurable if each of the corresponding entries is measurable.

Definition 2.5. Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Given a function g in $L^2(\mathbb{R})$, we associate with it the matrix-valued function $\mathcal{G} : \mathbb{R}^2 \to \mathcal{M}_{q,p}$ defined by

$$\mathcal{G}(t, v)_{r,k} = \left(\mathcal{Z}_{aq} g\right)\left(t + \frac{k}{b} - ra, v\right), \quad 0 \le r \le q-1, 0 \le k \le p-1, \tag{2.4}$$

for a.e. $(t, v) \in \mathbb{R}^2$.

The matrix-valued function $\mathcal{G}$ is related to the so-called Zibulski–Zeevi matrix [19] and has similar properties, but the definition given here is more convenient for our purposes. We will need the following lemma.

Lemma 2.6. *Let a, $b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$ and let S be a measurable subset of $\mathbb{R}$ which is a $\mathbb{Z}$-shift invariant. Given $g \in L^2(S)$, let, for a.e. $(t, v) \in \mathbb{R}^2$, $\mathcal{G}(t, v)$ be the matrix-valued function defined in (2.4) and let the matrix $\mathcal{P}(t, v) \in \mathcal{M}_{p,p}$ denote the orthogonal projection onto the kernel of $\mathcal{G}(t, v)$. Then, $\mathcal{P}(\cdot, \cdot)$ is measurable. Furthermore, the integer-valued function $(t, v) \mapsto \operatorname{rank}(\mathcal{G}(t, v))$ is measurable, $\frac{1}{bq}$-periodic with respect to variable t and satisfies the inequality*

$$\operatorname{rank}(\mathcal{G}(t, v)) \le \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right). \tag{2.5}$$

Proof. Note first that, for a.e. $(v, t) \in \mathbb{R}^2$,

$$\mathcal{P}(t, v) = \lim_{n \to \infty} \exp(-n(\mathcal{G}^* \mathcal{G})(t, v)),$$

by an easy application of the spectral theorem for self-adjoint matrices (see also [5, p. 978]). Since $\mathcal{G}(\cdot, \cdot)$ is measurable, the measurability of $\mathcal{P}(\cdot, \cdot)$, follows immediately. Using the facts that the sum of the rank of $\mathcal{G}(t, v)$ and the dimension of the kernel of $\mathcal{G}(t, v)$ is equal to p and that the dimension of a subspace of $\mathbb{C}^p$ is the trace of the orthogonal projection onto that subspace, it follows that the rank of $\mathcal{G}(t, v)$ is equal to $p - \operatorname{trace}(\mathcal{P}(t, v))$ and is thus also measurable. Given any $j \in \mathbb{Z}$, we can write, using Lemma 2.3, $\frac{j}{q} = k_0 + mp + r_0 \frac{p}{q}$ uniquely with $m \in \mathbb{Z}$, $k_0 \in \{0, 1, \ldots, p-1\}$ and $r_0 \in \{0, 1, \ldots, q-1\}$. If $k_1, k_2 \in \{0, 1, \ldots, p-1\}$, we have

$$\begin{aligned}(\mathcal{G}^* \mathcal{G})(t, v)_{k_1,k_2} &= \sum_{r=0}^{q-1} \overline{\mathcal{G}(t, v)_{r,k_1}} \mathcal{G}(t, v)_{r,k_2} \\ &= \sum_{r=0}^{q-1} \overline{(\mathcal{Z}_{aq} g)\left(t + \frac{k_1}{b} - ra, v\right)} (\mathcal{Z}_{aq} g)\left(t + \frac{k_2}{b} - ra, v\right).\end{aligned}$$

Hence,

$$
\begin{aligned}
&(\mathcal{G}^* \mathcal{G}) \left(t + \frac{j}{bq}, v\right)_{k_1,k_2} \\
&= \sum_{r=0}^{q-1} \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{j}{bq} + \frac{k_1}{b} - ra, v\right)} (\mathcal{Z}_{aq} g) \left(t + \frac{j}{bq} + \frac{k_2}{b} - ra, v\right) \\
&= \sum_{r=0}^{q-1} \left\{ \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} + \frac{mp}{b} - (r - r_0)a, v\right)} \right. \\
&\quad \left. \times\ (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_2}{b} + \frac{mp}{b} - (r - r_0)a, v\right) \right\}.
\end{aligned}
$$

Using Eq. (2.1) and the fact that $\frac{m\,p}{b} = m\,a\,q$, this expression simplifies to

$$
\sum_{r=0}^{q-1} \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} - (r - r_0)a, v\right)} (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_2}{b} - (r - r_0)a, v\right)
$$

or

$$
\begin{aligned}
&\sum_{r=0}^{r_0-1} \left\{ \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} - (r - r_0 + q)a + qa, v\right)} \right. \\
&\quad \left. \times\ (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_2}{b} - (r - r_0 + q)a + qa, v\right) \right\} \\
&\quad + \sum_{r=r_0}^{q-1} \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} - (r - r_0)a, v\right)} (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_2}{b} - (r - r_0)a, v\right).
\end{aligned}
$$

Using again Eq. (2.1), we can rewrite this last expression as

$$
\sum_{r=0}^{q-1} \overline{(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} - ra, v\right)} (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_2}{b} - ra, v\right).
$$

Using the fact that

$$
(\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1}{b} - ra, v\right) = \mathrm{e}^{-2\pi \mathrm{i} v} (\mathcal{Z}_{aq} g) \left(t + \frac{k_0 + k_1 - p}{b} - ra, v\right)
$$

it follows thus that, for $k_1, k_2 \in \{0, \ldots, p-1\}$, the entry $(\mathcal{G}^* \mathcal{G}) \left(t + \frac{j}{bq}, v\right)_{k_1,k_2}$ must be equal to

$$
\begin{cases}
(\mathcal{G}^* \mathcal{G})(t, v)_{k_1+k_0, k_2+k_0}, & \text{if } k_1 + k_0 < p \text{ and } k_2 + k_0 < p, \\
\mathrm{e}^{-2\pi \mathrm{i} v} (\mathcal{G}^* \mathcal{G})(t, v)_{k_1+k_0, k_2+k_0-p}, & \text{if } k_1 + k_0 < p \text{ and } k_2 + k_0 \geq p, \\
\mathrm{e}^{2\pi \mathrm{i} v} (\mathcal{G}^* \mathcal{G})(t, v)_{k_1+k_0-p, k_2+k_0}, & \text{if } k_1 + k_0 \geq p \text{ and } k_2 + k_0 < p, \\
(\mathcal{G}^* \mathcal{G})(t, v)_{k_1+k_0-p, k_2+k_0-p}, & \text{if } k_1 + k_0 \geq p \text{ and } k_2 + k_0 \geq p.
\end{cases}
$$

If $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_{p-1})^t \in \mathbb{C}^p$, define $U\boldsymbol{\xi} = \boldsymbol{\eta} = (\eta_0, \ldots, \eta_{p-1})^t$, where

$$
\eta_i = \begin{cases}
\mathrm{e}^{-2\pi \mathrm{i} v}\, \xi_{i-k_0+p}, & \text{if } 0 \leq i \leq k_0 - 1, \\
\xi_{i-k_0}, & \text{if } k_0 \leq i \leq p - 1.
\end{cases}
$$

Then, U is a $p \times p$ unitary matrix and

$$\left\langle (\mathcal{G}^* \mathcal{G}) \left(t + \frac{j}{bq}, v \right) \boldsymbol{\xi}, \boldsymbol{\xi} \right\rangle = \langle (\mathcal{G}^* \mathcal{G})(t, v) \boldsymbol{\eta}, \boldsymbol{\eta} \rangle = \langle (\mathcal{G}^* \mathcal{G})(t, v) U \boldsymbol{\xi}, U \boldsymbol{\xi} \rangle$$
$$= \langle U^* (\mathcal{G}^* \mathcal{G})(t, v) U \boldsymbol{\xi}, \boldsymbol{\xi} \rangle,$$

which shows that $(\mathcal{G}^* \mathcal{G})(t + \frac{j}{bq}, v) = U^* (\mathcal{G}^* \mathcal{G})(t, v) U$ and thus $(\mathcal{G}^* \mathcal{G})(t + \frac{j}{bq}, v)$ and $(\mathcal{G}^* \mathcal{G})(t, v)$ must have the same rank. Since the rank of any matrix A is the same as that of $A^* A$, it follows that $\operatorname{rank}(\mathcal{G}(t, v))$ is $\frac{1}{bq}$-periodic with respect to variable t. Finally, it follows from Lemma 2.1 that $(\mathcal{Z}_{aq} g)(t + \frac{k}{b} - ra, v) = 0$ if $\chi_S(t + \frac{k}{b}) = 0$ so that a column of $(\mathcal{G})(t, v)$ corresponding to an index k such that $\chi_S(t + \frac{k}{b}) = 0$ must be identically zero. The rank of $(\mathcal{G})(t, v)$ is then at most equal to the numbers of the other columns which is $\sum_{k=0}^{p-1} \chi_S \left(t + \frac{k}{b} \right)$. This proves the lemma. □

The following result provides a characterization for the completeness of the span of Gabor system in $L^2(S)$ in terms of the matrix-valued function $\mathcal{G}$ associated with the window.

Theorem 2.7. *Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Let S be a measurable set in $\mathbb{R}$ with nonzero measure and suppose that S is a $\mathbb{Z}$-shift invariant. Assume that $g \in L^2(S)$ and let $\mathcal{G}(t, v)$ denote the $q \times p$ matrix-valued function defined by (2.4). Then the following are equivalent:*

(a) *The linear span of the collection $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$.*
(b) $\operatorname{Rank}(\mathcal{G}(t, v)) = \sum_{k=0}^{p-1} \chi_S \left(t + \frac{k}{b} \right)$ *for a.e.* $(t, v) \in [0, \frac{1}{bq}) \times (0, 1)$.
(c) $\operatorname{Rank}(\mathcal{G}(t, v)) = \sum_{k=0}^{p-1} \chi_S \left(t + \frac{k}{b} \right)$ *for a.e.* $(t, v) \in \mathbb{R}^2$.

Proof. The equivalence of (b) and (c) follows from the fact that the functions on either side of the equality in (c) are $\frac{1}{bq}$-periodic with respect to the first variable t, by Lemmas 2.4 and 2.6, and are also clearly 1-periodic with respect to the second variable v. Define $g_r(\cdot) = g(\cdot - ra)$ for $r = 0, 1, \ldots, q - 1$. Then, the linear span of the collection $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$ if and only if the span of $\left\{ E_{mb} T_{naq} g_r : 0 \le r \le q - 1, m, n \in \mathbb{Z} \right\}$ is dense in $L^2(S)$. It is easy to check that

$$\left(\mathcal{Z}_{aq} E_{mb} T_{naq} g_r \right)(t, v) = \mathrm{e}^{2\pi \mathrm{i} mbt} \mathrm{e}^{2\pi \mathrm{i} nv} \left(\mathcal{Z}_{aq} g \right)(t - ra, v)$$

for $m, n \in \mathbb{Z}$ and $0 \le r \le q - 1$. Applying Lemma 2.1, we have, for $f \in L^2(\mathbb{R})$, $m, n \in \mathbb{Z}$ and $0 \le r \le q - 1$,

$$\langle f, E_{mb} T_{naq} g_r \rangle = \int_0^{aq} \int_0^1 \left(\mathcal{Z}_{aq} f \right)(t, v) \overline{\left(\mathcal{Z}_{aq} g \right)(t - ra, v)} \mathrm{e}^{-2\pi \mathrm{i} mbt} \mathrm{e}^{-2\pi \mathrm{i} nv} \mathrm{d}v \mathrm{d}t$$
$$= \int_0^{\frac{1}{b}} \int_0^1 \sum_{k=0}^{p-1} \left(\mathcal{Z}_{aq} f \right) \left(t + \frac{k}{b}, v \right) \overline{\left(\mathcal{Z}_{aq} g \right) \left(t + \frac{k}{b} - ra, v \right)}$$
$$\times \mathrm{e}^{-2\pi \mathrm{i} mbt} \mathrm{e}^{-2\pi \mathrm{i} nv} \mathrm{d}v \mathrm{d}t. \tag{2.6}$$

If (c) holds, let $f \in L^2(S)$ satisfy that $\langle f, E_{mb} T_{naq} g_r \rangle = 0$ for all $m, n \in \mathbb{Z}$ and $0 \le r \le q - 1$. We need to prove that $f = 0$. For fixed $(t, v) \in \mathbb{R}^2$, let $\mathtt{F}(t, v) = (F_0(t, v), \ldots, F_{p-1}(t, v))^t \in \mathbb{C}^p$ be defined by $F_i(t, v) = \overline{\left(\mathcal{Z}_{aq} f \right)}(t + \frac{i}{b}, v)$ for $i = 0, \ldots, p - 1$. By (2.6), we have

$$\mathcal{G}(t, v) \mathtt{F}(t, v) = 0, \quad \text{for a.e. } (t, v) \in \left[0, \frac{1}{b} \right) \times [0, 1). \tag{2.7}$$

If B is a subset of $\{0, 1, \ldots, p-1\}$, we define

$$I_B = \left\{ t \in \left[0, \frac{1}{b}\right), \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) = \sum_{k \in B} \chi_S\left(t + \frac{k}{b}\right) = \mathrm{card}(B) \right\}.$$

Then, each set I_B is measurable and the collection $\{I_B\}$, where B runs over all subsets of $\{0, 1, \ldots, p-1\}$, forms a partition of the interval $[0, \frac{1}{b})$. If $B = \emptyset$ and $t \in I_\emptyset$, we have $\sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) = 0$ and thus $\mathtt{F}(t, v) = 0$ by Lemma 2.1. If $B \neq \emptyset$ and $t \in I_B$, we have $\mathcal{G}(t, v)_{r,k} = F_k(t, v) = 0$ if $k \notin B$, by Lemma 2.1, since f and $g_r \in L^2(S)$, for each $r = 0, \ldots, q-1$. Using our assumption, the submatrix of $\mathcal{G}(t, v)$ of size $q \times \mathrm{card}(B)$ obtained by removing from $\mathcal{G}(t, v)$ all the columns with an index not in B has thus a rank equal to $\mathrm{card}(B)$, since the entries corresponding to the removed columns are all zero, and Eq. (2.7) then implies that $F_k(t, v) = 0$, for $k \in B$. Hence, $\mathtt{F}(t, v) = 0$ if $t \in I_B$. Therefore, $\mathtt{F}(t, v) = 0$ for $t \in [0, \frac{1}{b})$, which shows that $\left(\mathcal{Z}_{aq} f\right)(t, v) = 0$ for $t \in [0, \frac{p}{b}) = [0, aq)$ and thus that $f = 0$, using Lemma 2.1 again.

Conversely, if (c), or equivalently, (b) fails, then, taking into account inequality (2.5), we deduce the existence of a subset B of $\{0, 1, \ldots, p-1\}$ such that

$$\mathrm{Rank}(\mathcal{G}(t, v)) < \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right), \quad \text{for a.e. } (t, v) \in H, \tag{2.8}$$

where H is a measurable subset of $I_B \times [0, 1)$ with nonzero measure. Let $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{p-1}$ denote the standard orthonormal basis of $\mathbb{C}^p$ and let $\mathcal{P}(t, v) : \mathbb{C}^p \to \mathbb{C}^p$ denote the orthogonal projection onto the kernel of $\mathcal{G}(t, v)$. Then, $\mathcal{P}(\cdot, \cdot)$ is measurable by Lemma 2.6. We claim that there exists $k_0 \in B$ such that the vector-valued function $\mathcal{P}(t, v)\mathbf{e}_{k_0} \neq 0$ on a subset of $I_B \times [0, 1)$ having positive measure. Indeed if this were not the case, letting $E = \mathrm{span}\{\mathbf{e}_k : k \in B\}$, it would follow that for a.e. $(t, v) \in I_B \times [0, 1)$, $\mathcal{P}(t, v)\mathbf{x} = 0$, for all $\mathbf{x} \in E$ or, equivalently, that $E \oplus \ker(\mathcal{G}(t, v))$ is a direct sum. Since, in that case,

$$p \geq \dim(E \oplus \ker(\mathcal{G}(t, v))) = \mathrm{card}(E) + (p - \mathrm{rank}(\mathcal{G}(t, v))),$$

this would imply that $\mathrm{rank}(\mathcal{G}(t, v)) \geq \mathrm{card}(E)$ and thus, using the definition of I_B, that

$$\mathrm{rank}(\mathcal{G}(t, v)) = \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) \quad \text{for a.e. } (t, v) \in I_B \times [0, 1).$$

This would contradict (2.8). With $k_0 \in B$ as above, we define

$$\mathtt{F}(t, v) = \begin{cases} \mathcal{P}(t, v)\mathbf{e}_{k_0}, & \text{if } (t, v) \in I_B \times [0, 1), \\ 0, & \text{if } (t, v) \in \left(\left[0, \frac{1}{b}\right) \setminus I_B\right) \times [0, 1). \end{cases}$$

By construction, we have $\|\mathtt{F}(t, v)\|_{\mathbb{C}^p} \leq 1$, so that each component of $\mathtt{F}$ is square-integrable on $[0, \frac{1}{b}) \times [0, 1)$ and $\mathtt{F} = (F_0, \ldots, F_{p-1})^t$ satisfies Eq. (2.7). Furthermore, if $l \in \{0, 1, \ldots, p-1\} \setminus B$ and $(t, v) \in I_B \times [0, 1)$, we have that $\mathcal{G}(t, v)\mathbf{e}_l = 0$ and thus

$$\langle \mathcal{P}(t, v)\mathbf{e}_{k_0}, \mathbf{e}_l \rangle = \langle \mathbf{e}_{k_0}, \mathcal{P}(t, v)\mathbf{e}_l \rangle = \langle \mathbf{e}_{k_0}, \mathbf{e}_l \rangle = 0.$$

This shows that, if $(t, v) \in [0, \frac{1}{b}) \times [0, 1)$, we must have $F_k(t, v) = 0$ whenever $\chi_S\left(t + \frac{k}{b}\right) = 0$. Defining $f \in L^2(\mathbb{R})$ by

$$\left(\mathcal{Z}_{aq} f\right)\left(t + \frac{k}{b}, v\right) = F_k(t, v), \quad (t, v) \in \left[0, \frac{1}{b}\right) \times [0, 1),\ k = 0, 1, \ldots, p-1,$$

we have $f \neq 0$ and $\{\mathcal{Z}_{aq} f \neq 0\} \subset S_0 \times [0, 1)$. Hence, f belongs to $L^2(S)$ by Lemma 2.1, and, furthermore, using (2.7), f is orthogonal to the collection $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$. Hence, (a) fails which completes the proof. $\square$

In the case, $S = \mathbb{R}$, the density condition in the theorem just proved reduces to $\operatorname{rank}(\mathcal{G}) = p$ a.e., a condition which also must hold for the Zibulski–Zeevi matrix ([5,19]; see also [12]). As in the case $S = \mathbb{R}$, this result has an important consequence.

Corollary 2.8. *Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$ and let S be an $a\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$. If there exists a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$, then, necessarily*

$$\sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) \leq q \quad \text{for a.e. } t \in \mathbb{R}. \tag{2.9}$$

Proof. By the previous theorem, we have, for a.e. $t \in \mathbb{R}$,

$$\sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) = \operatorname{Rank}(\mathcal{G}(t, v)) \leq q,$$

since $\mathcal{G}(t, v) \in \mathcal{M}_{q,p}$. $\square$

Our next goal will be to prove that in the rational case ($ab \in \mathbb{Q}$), condition (2.9) is sufficient to ensure the existence of a function $g \in L^2(S)$ such that the collection $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ not only has a dense linear span in $L^2(S)$, but forms a tight frame for $L^2(S)$. In fact, we will see that this can be done with g of the form $g = \chi_E$, where E is a subset of S such that $\chi_S = \sum_{n \in \mathbb{Z}} \chi_E(\cdot - n\,a)$. On the other hand, if E is a set satisfying the previous identity and $g = \chi_E$, it is clear that $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ forms a tight frame for $L^2(S)$ if and only if $\{E_{mb} \chi_E : m \in \mathbb{Z}\}$ is a tight frame for $L^2(E)$. The next lemma translates this last requirement into geometrical terms. We first need the following definition.

Definition 2.9. Given $a > 0$, two measurable subsets of $\mathbb{R}$, E_1 and E_2, are said to be $a\,\mathbb{Z}$-*congruent* if there exist measurable partitions $\{E_{1,k}\}_{k \in \mathbb{Z}}$ of E_1 and $\{E_{2,k}\}_{k \in \mathbb{Z}}$ of E_2 such that $E_{2,k} = E_{1,k} + k\,a$, modulo a set of zero measure, for all $k \in \mathbb{Z}$. It is easy to check that E_1 and E_2 are $a\,\mathbb{Z}$-translation congruent if and only if the identity

$$\sum_{k \in \mathbb{Z}} \chi_{E_1}(\cdot - a\,k) = \sum_{k \in \mathbb{Z}} \chi_{E_2}(\cdot - a\,k)$$

holds a.e. on $\mathbb{R}$.

Lemma 2.10. *Let $b > 0$, and E be a measurable subset of $\mathbb{R}$. Then, the following conditions are equivalent:*

(a) $\{E_{mb} \chi_E : m \in \mathbb{Z}\}$ *is a frame for* $L^2(E)$.

(b) *The linear span of the collection* $\{E_{mb}\chi_E : m \in \mathbb{Z}\}$ *is dense in* $L^2(E)$.
(c) E *is* $\frac{1}{b}\mathbb{Z}$*-congruent to a subset of* $[0, \frac{1}{b})$.
(d) $\sum_{k\in\mathbb{Z}} \chi_E\left(\cdot + \frac{k}{b}\right) \le 1$ *a.e. on* $\mathbb{R}$.

In addition, $\{E_{mb}\chi_E : m \in \mathbb{Z}\}$ *is a tight frame in* $L^2(E)$ *with frame bound* $\frac{1}{b}$ *if any of the above conditions holds.*

Proof. It is clear that (a) implies (b). To show that (b) implies (c), assume that (b) holds and define $E_l = E \cap [l/b, (l+1)/b)$ for $l \in \mathbb{Z}$. Then $\{E_l : l \in \mathbb{Z}\}$ is a partition of E and $\bigcup_{l\in\mathbb{Z}}(E_l - l/b) \subset [0, \frac{1}{b})$. So it suffices to prove that $\left|(E_l - l/b)\bigcap(E_k - k/b)\right| = 0$ for $l \ne k$, $l, k \in \mathbb{Z}$. If this were not the case, there would exist $l_0, k_0 \in \mathbb{Z}$ with $l_0 \ne k_0$ such that $F := \left(E_{l_0} - l_0/b\right)\bigcap\left(E_{k_0} - k_0/b\right)$ has positive measure. Define $f \in L^2(E)$ by

$$f = \begin{cases} 1 & \text{on } F + l_0/b; \\ -1 & \text{on } F + k_0/b; \\ 0 & \text{on } E \setminus [(F + l_0/b) \cup (F + k_0/b)]. \end{cases}$$

Then, for $m \in \mathbb{Z}$,

$$\int_E f(x)\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x = \int_{F+l_0/b} [f(x) + f(x + (k_0 - l_0)/b)]\,\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x = 0$$

contradicting the fact that the linear span of the collection $\{E_{mb}\chi_E : m \in \mathbb{Z}\}$ is dense in $L^2(E)$.

The equivalence of (c) and (d) is clear. To finish the proof, we show that (c) implies that $\{E_{mb}\chi_E : m \in \mathbb{Z}\}$ is a tight frame in $L^2(E)$ with frame bound $\frac{1}{b}$ and thus also statement (a). Suppose $\{E_l : l \in \mathbb{Z}\}$ is a partition of E such that $\{E_l - l/b : l \in \mathbb{Z}\}$ is a partition of some subset of $[0, \frac{1}{b})$. Then, for $f \in L^2(E)$, we have

$$\begin{aligned} \langle f, E_{mb}\chi_E\rangle &= \sum_{l\in\mathbb{Z}} \int_{E_l - l/b} f(x - l/b)\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x \\ &= \int_{[0,\frac{1}{b})} \sum_{l\in\mathbb{Z}} f(x - l/b)\,\chi_{E_l - l/b}(x)\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x \end{aligned}$$

and, consequently, using Parseval's formula,

$$\begin{aligned} \sum_{m\in\mathbb{Z}} |\langle f, E_{mb}\chi_E\rangle|^2 &= \frac{1}{b}\int_{[0,\frac{1}{b})} \left|\sum_{l\in\mathbb{Z}} f(x - l/b)\chi_{E_l - l/b}(x)\right|^2 \mathrm{d}x \\ &= \frac{1}{b}\sum_{l\in\mathbb{Z}} \int_{E_l - l/b} |f(x - l/b)|^2\,\mathrm{d}x = \frac{1}{b}\int_E |f(x)|^2\,\mathrm{d}x, \end{aligned}$$

which completes the proof. $\square$

In connection with the previous lemma, we mention the following particular case of a well-known result about spectral pairs due to Fuglede [9].

Proposition 2.11. *Let* $b > 0$*, and* E *be a measurable subset of* $\mathbb{R}$*. Then, the following conditions are equivalent:*

(a) $\{\sqrt{b}\,E_{mb}\chi_E : m \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(E)$.
(b) $\sum_{k\in\mathbb{Z}} \chi_E\left(\cdot + \frac{k}{b}\right) = 1$ *a.e. on* $\mathbb{R}$.

Theorem 2.12. *Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Let S be a measurable subset of $\mathbb{R}$ which is a $\mathbb{Z}$-shift invariant. Then, the following are equivalent.*

(i) *There exists a measurable set E in $\mathbb{R}$ which is $a\,\mathbb{Z}$-congruent to $S \cap [0, a)$ such that $\{E_{mb}T_{na}\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.*

(ii) *We have the inequality*

$$\sum_{k=0}^{p-1} \chi_S\left(\cdot + \frac{k}{b}\right) \le q \quad \text{a.e. on } \mathbb{R}. \tag{2.10}$$

Proof. The necessity of condition (2.10) is a direct consequence of Corollary 2.8. It can also be obtained by the following, more direct, observation. The facts that E is $a\,\mathbb{Z}$-congruent to $S \cap [0, a)$ and that $\{E_{mb}T_{na}\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$ are equivalent, by Lemma 2.10, to

$$\sum_{k\in\mathbb{Z}} \chi_E(\cdot + k\,a) = \chi_S \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \chi_E(\cdot + k/b) \le 1 \quad \text{a.e. on } \mathbb{R},$$

respectively. Therefore, we have

$$\begin{aligned}\sum_{k=0}^{p-1} \chi_S\left(\cdot + \frac{k}{b}\right) &= \sum_{k=0}^{p-1}\sum_{l\in\mathbb{Z}} \chi_E\left(\cdot + \frac{l\,p}{q\,b} + \frac{k}{b}\right) = \sum_{j\in\mathbb{Z}} \chi_E\left(\cdot + \frac{j}{q\,b}\right) \\ &= \sum_{r=0}^{q-1}\sum_{k\in\mathbb{Z}} \chi_E\left(\cdot + \frac{r}{q\,b} + \frac{k}{b}\right) \le q.\end{aligned}$$

Let $S_0 = S \cap [0, a)$. Clearly, $\{S_0 + na : n \in \mathbb{Z}\}$ is a partition of S. To prove the sufficiency part of the statement, we need to show, according to Lemma 2.10, that there exists a measurable set E in $\mathbb{R}$ such that E is $a\,\mathbb{Z}$-congruent to S_0 and at the same time is $\frac{1}{b}\mathbb{Z}$-congruent to a subset of $[0, 1/b)$. If $ab \le 1$, we can take $E = S_0$ since $S_0 \subset [0, a) \subset [0, 1/b)$ in this case. We can also reduce the proof of the construction of E to the case $b = 1$. Indeed, if b is arbitrary, we can define $\check{S} = bS$. Then $\check{S}$ is $a\,b\,\mathbb{Z}$-shift invariant, and $\left\{\check{S}_0 + nab : n \in \mathbb{Z}\right\}$ is a partition of $\check{S}$, where $\check{S}_0 = \check{S} \cap [0, ab)$. Furthermore, $\check{S}$ satisfies (2.10) with b replaced by 1. So, if we can construct a measurable set $\check{E}$ such that $\check{E}$ is $a\,b\,\mathbb{Z}$-congruent to $\check{S}_0$, and with $\check{E}$ being $\mathbb{Z}$-congruent to a subset of $[0, 1)$, we can then define $E = \frac{1}{b}\check{E}$. We can easily check that E satisfies all of our requirements. We may thus assume, without loss of generality, that $b = 1$ and $a > 1$. We have $a = \frac{p}{q}$, and, using (2.3) with $b = 1$ and (2.10), it follows that

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = \sum_{k\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{k}{q}\right) \le q. \tag{2.11}$$

Note that, since

$$|S_0| = \int_{[0,\frac{1}{q})} \sum_{k\in\mathbb{Z}} \chi_{S_0}(x + k/q)\,\mathrm{d}x,$$

inequality (2.11) implies that $|S_0| \le 1$ as well as the fact that $|S_0| = 1$ if and only if

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = q \quad \text{a.e. on } \mathbb{R}. \tag{2.12}$$

We will divide the proof into two cases: $|S_0| = 1$ and $|S_0| < 1$.

Case 1: $|S_0| = 1$.

In this case, identity (2.12) holds. Let $g_0 = \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k)$. If $g_0 \le 1$ a.e. on $\mathbb{R}$, then $E = S_0$ is as desired. Otherwise, we follow the following inductive procedure to construct E. Let $H_0 = \{g_0 > 0\}$, where $\{g_0 > 0\}$ denotes the set $\{t \in \mathbb{R} : g_0(t) > 0\}$. (We will use similar notations for the sets H_k defined below.) Let T_0 be a measurable subset of S_0 such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_0}(\cdot + k) = \chi_{H_0}.$$

If the sets S_l, T_l, H_l and the function g_l have already been constructed for all indices l with $0 \le l \le j-1 < q-1$, we define $S_j = S_{j-1} \setminus T_{j-1}$,

$$g_j = \sum_{k\in\mathbb{Z}} \chi_{S_j}(\cdot + a\,j + k),$$

and $H_j = \{g_j > 0,\ g_0 = g_1 = \cdots = g_{j-1} = 0\}$. We then choose a measurable set T_j contained in S_j and such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_j}(\cdot + a\,j + k) = \chi_{H_j}.$$

We follow this procedure until the index j above reaches $q-1$ and then stop. We then define $E = \bigcup_{i=0}^{q-1} (T_i - ia)$. Note that the sets H_i, $i = 0, \ldots, q-1$, are mutually disjoint. We have thus

$$\sum_{k\in\mathbb{Z}} \chi_E(\cdot + k) = \sum_{i=0}^{q-1} \sum_{k\in\mathbb{Z}} \chi_{T_i}(\cdot + \mathrm{i}\,a + k) = \sum_{i=0}^{q-1} \chi_{H_i} \le 1 \quad \text{a.e. on } \mathbb{R}.$$

So, by Lemma 2.10, the collection $\left\{e^{2\pi \mathrm{i} mt}\, \chi_E(t) : m \in \mathbb{Z}\right\}$ is a tight frame for $L^2(E)$. Since the sets T_i, $i = 0, \ldots, q-1$ are disjoint subsets of $S_0 \subset [0, a)$, in order to show that E is $a\,\mathbb{Z}$-congruent to S_0, we only need to prove that $\sum_{i=0}^{q-1} |T_i| = |S_0| = 1$. We have

$$\sum_{i=0}^{q-1} |T_i| = \sum_{i=0}^{q-1} \int_{[0,1)} \sum_{k\in\mathbb{Z}} \chi_{T_i}(t + \mathrm{i}\,a + k)\mathrm{d}t = \int_{[0,1)} \sum_{i=0}^{q-1} \chi_{H_i}(t)\mathrm{d}t$$

and since $\sum_{i=0}^{q-1} \chi_{H_i} \le 1$ a.e. on $\mathbb{R}$, it suffices to prove that $\sum_{i=0}^{q-1} \chi_{H_i} = 1$ a.e. on $[0, 1)$. We will argue by contradiction. Suppose that there exists a measurable set $F \subset [0, 1)$ with nonzero measure such that $g_i = 0$ on F for all indices i with $0 \le i \le q-1$. If $q = 1$, then $\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = g_0 = 0$ on F, contradicting the fact that $\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = 1$ a.e. on $\mathbb{R}$ (which follows from (2.11) and (2.12)). If $q > 1$, we have

$$g_0 = \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = 0 \quad \text{on } F$$

and, since $\chi_{S_l} = \chi_{S_0} - \sum_{i=0}^{l-1} \chi_{T_i}$, the fact that $g_l = 0$ on F for $1 \le l \le q-1$ is equivalent to

$$\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + l\,a + k) = \sum_{i=0}^{l-1} \sum_{k\in\mathbb{Z}} \chi_{T_i}(\cdot + l\,a + k) = \sum_{i=0}^{l-1} \chi_{H_i}(\cdot + (l-i)a)$$

on F for $l = 1, 2, \ldots, q-1$. Also observing that

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = \sum_{r=0}^{q-1} \sum_{k=0}^{p-1} \sum_{l\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{(lq+r)p}{q} + k\right) = \sum_{r=0}^{q-1} \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + ra + k),$$

we have

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = \sum_{r=1}^{q-1}\sum_{i=0}^{r-1} \chi_{H_i}(\cdot + (r-i)a) = \sum_{l=1}^{q-1}\sum_{i=0}^{q-l-1} \chi_{H_i}(\cdot + l\,a) \le q-1 < q$$

on F, which contradicts (2.12).

Case 2: $|S_0| < 1$.

For $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{p-1}) \in \{0,1\}^p$, define

$$A(\boldsymbol{\epsilon}) = \left\{ t \in [0, 1/q) : \chi_{S_0}\left(t + \frac{k}{q}\right) = \epsilon_k \text{ for } 0 \le k \le p-1 \right\}.$$

If $\sum_{k=0}^{p-1} \epsilon_k = m < q$, we choose $m - q$ indices k_i, $i = 1, \ldots, q-m$, such that $\epsilon_{k_1} = \epsilon_{k_2} = \cdots = \epsilon_{k_{q-m}} = 0$, which can be done since $p > q$, and we define

$$B(\boldsymbol{\epsilon}) = \left(\bigcup_{l=1}^{q-m}\left(A(\boldsymbol{\epsilon}) + \frac{k_l}{q}\right)\right) \bigcup \left(\bigcup_{\epsilon_k \neq 0}\left(A(\boldsymbol{\epsilon}) + \frac{k}{q}\right)\right).$$

If $\sum_{k=0}^{p-1} \epsilon_k = q$, define

$$B(\boldsymbol{\epsilon}) = \bigcup_{\epsilon_k \neq 0}\left(A(\boldsymbol{\epsilon}) + \frac{k}{q}\right).$$

We then let

$$\tilde{S}_0 = \bigcup_{\boldsymbol{\epsilon}\in\{0,1\}^p} B(\boldsymbol{\epsilon}), \quad \text{and} \quad \tilde{S} = \bigcup_{n\in\mathbb{Z}}\left(\tilde{S}_0 + na\right).$$

Note that, by construction, $S_0 \subset \tilde{S}_0 \subset [0, a)$. Furthermore, we have

$$\sum_{k\in\mathbb{Z}} \chi_{\tilde{S}_0}\left(\cdot + \frac{k}{q}\right) = \sum_{k=0}^{p-1} \chi_{\tilde{S}}(\cdot + k) = q \quad \text{a.e. on } \mathbb{R},$$

which implies, as before that $|\tilde{S}_0| = 1$. Using Case 1, with S and S_0 replaced with $\tilde{S}$ and $\tilde{S}_0$, respectively, we can construct a measurable set $\tilde{E}$ which is $a\,\mathbb{Z}$-congruent to $\tilde{S} \cap [0, a)$ and such that $\left\{e^{2\pi imt}\chi_{\tilde{E}}(t) : m \in \mathbb{Z}\right\}$ is a tight frame for $L^2(\tilde{E})$. The collection $\left\{e^{2\pi imt}\chi_{\tilde{E}}(t - n\,a) : m, n \in \mathbb{Z}\right\}$ is thus a tight frame for $L^2(\tilde{S})$ and the set $E := \tilde{E} \cap S$ satisfies all of our requirements. $\square$

Corollary 2.13. *Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$ and let S be an $a\,\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ and define the set $S_0 = S \cap [0, a)$. Then, the following are equivalent:*

(a) *There exists a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$.*

(b) *There exists a measurable set E in $\mathbb{R}$ which is $a\,\mathbb{Z}$-congruent to $S \cap [0, a)$ such that $\{E_{mb}T_{na}\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.*

(c) $\sum_{k=0}^{p-1} \chi_S\left(\cdot + \frac{k}{b}\right) \le q$ *a.e. on* $\mathbb{R}$.

In particular, if any of the conditions above holds, then we must have the inequality

$$b\,|S_0| \le 1. \tag{2.13}$$

Proof. The equivalence between statements (a), (b) and (c) follows immediately from Corollary 2.8 and Theorem 2.12. If (c) holds, then we must have

$$\int_0^{1/b} \sum_{k=0}^{p-1} \chi_S\left(t+\frac{k}{b}\right) \mathrm{d}t = \int_0^{p/b} \chi_S(t)\mathrm{d}t = \int_0^{a\,q} \chi_S(t)\mathrm{d}t = q\,|S_0| \le \frac{q}{b},$$

which yields inequality (2.13). □

It was proved in [11] that the existence of a function $g \in L^2(S)$ such that the linear span of $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$ implies inequality (2.13), even without the restriction that $a\,b \in \mathbb{Q}$. In the rational case, it is clear that condition (2.10) is always stronger than condition (2.13) when $a\,b = p/q > 1$. (Note that both conditions are always clearly satisfied when $a\,b = p/q \le 1$.) For example, if we define, for $0 < \epsilon < \min(\frac{1}{b\,q}, \frac{1}{p})$, the set

$$S = \bigcup_{l\in\mathbb{Z}} \left\{ \bigcup_{k=0}^{p-1} \left[\frac{k}{b}, \frac{k}{b} + \epsilon\right) + l\,a \right\},$$

we have $b\,|S_0| = \epsilon\,p \le 1$, but condition (2.10) clearly fails when $p > q$. However, in the irrational case, condition (2.13) turns out to be necessary and sufficient for the existence of a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is dense (or, forms a tight frame) in $L^2(S)$, as we will prove in the last section.

Remark. In the rational case, if we consider two Gabor systems associated with a fixed set S and with the same parameter a but with a different parameter b, say b_1 and b_2, the fact that density condition (2.10) holds for the pair (a, b_1) does not imply that it also holds for the pair (a, b_2) when $b_2 < b_1$. For example, if we choose $a = 1$, $b_1 = 5/2$, $b_2 = 2$ and

$$S = \bigcup_{n\in\mathbb{Z}} \left([n, 1/10 + n) \cup [5/10 + n, 6/10 + n)\right),$$

condition (2.10) holds for the pair $(a, b_1) = (1, 5/2)$ since

$$\sum_{k=0}^{4} \chi_S\left(\cdot + \frac{2\,k}{5}\right) = 1 \le 2,$$

while it does not for the pair $(a, b_2) = (1, 2)$ in view of the fact that

$$\sum_{k=0}^{1} \chi_S\left(\cdot + \frac{k}{2}\right) = 2 > 1$$

on the interval $(0, \frac{1}{10})$.

3. A necessary condition for the existence of Gabor subspace frames in $L^2(S)$

In this section, we provide a simple proof that the condition $b\,|S \cap [0, a)| \le 1$ is necessary, in both the rational and irrational cases, for the existence of a window such that the associated Gabor system with parameters a, b forms a frame for $L^2(S)$. Although the necessity of this condition was obtained earlier in [11, Corollary 2.4], the proof given there, based on methods of operator algebras, is less transparent. Furthermore, we show that, if such a system exists, it will be a Riesz basis for $L^2(S)$ if and only if equality occurs in the condition above. We first need the following lemmas. The first one of these is well-known [4, Proposition 2.1].

Lemma 3.1. *If $g \in L^\infty(\mathbb{R})$ is compactly supported, there is a constant C such that*

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb} T_{na}\, g\rangle|^2 \le C\, \|f\|_2^2, \quad f \in L^2(\mathbb{R}).$$

The second lemma deals with a version of the Walnut representation which was proved in [13, Proposition 7.1.1] under the assumption that the collections $\{E_{mb} T_{na}\, g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb} T_{na}\, \gamma\}_{m,n\in\mathbb{Z}}$ are both Bessel sequences. As we will show here, these conditions on g and γ are not necessary.

Lemma 3.2. *Let $a, b > 0$, and $g, \gamma \in L^2(\mathbb{R})$. Then, for $f, h \in L^\infty(\mathbb{R})$ with bounded support,*

$$\sum_{m,n\in\mathbb{Z}} \langle f, E_{mb} T_{na}\, g\rangle\, \langle E_{mb} T_{na}\gamma, h\rangle = \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} G_n(x) f\,(x - n/b)\,\overline{h(x)}\mathrm{d}x, \tag{3.1}$$

where $G_n = \sum_{k\in\mathbb{Z}} \overline{g(\cdot - \frac{n}{b} - k\,a)}\,\gamma(\cdot - k\,a)$ and both series in (3.1) converge absolutely.

Proof. By a simple computation, we have

$$\langle f, E_{mb} T_{na}\, g\rangle = \int_{[0,\frac{1}{b})} \sum_{l\in\mathbb{Z}} (T_{na}\overline{g}\, f)\,(x - l/b)\,\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x,$$

$$\langle h, E_{mb}T_{na}\gamma\rangle = \int_{[0,\frac{1}{b})} \sum_{l\in\mathbb{Z}} (T_{na}\overline{\gamma}\, h)\,(x - l/b)\,\mathrm{e}^{-2\pi\mathrm{i}mbx}\mathrm{d}x.$$

Also, observing that the sum of the series in both integrals above define functions in $L^2([0, \frac{1}{b}))$ since both f and h have bounded support, we have, by Parseval's formula and the fact that both f and h are of bounded support,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \langle f, E_{mb} T_{na}\, g\rangle\, \langle E_{mb} T_{na}\, \gamma, h\rangle \\
&\quad = \frac{1}{b}\int_{[0,\frac{1}{b})} \left[\sum_{l_1\in\mathbb{Z}} (T_{na}\overline{g}\, f)\,(x - l_1/b)\right] \left[\sum_{l_2\in\mathbb{Z}} \left(T_{na}\gamma\overline{h}\right)(x - l_2/b)\right] \mathrm{d}x \\
&\quad = \frac{1}{b}\int_{\mathbb{R}} \left(\sum_{l\in\mathbb{Z}} (T_{na}\overline{g}\, f)\,(x - l/b)\right) T_{na}\,\gamma(x)\overline{h(x)}\mathrm{d}x \\
&\quad = \frac{1}{b}\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \overline{g\,(x - na - l/b)}\,\gamma\,(x - na)\, f\,(x - l/b)\,\overline{h(x)}\mathrm{d}x.
\end{aligned}$$

Identity (3.1) follows immediately. The boundedness of the support of both f and h shows that the series on the right-hand side of (3.1) converges absolutely since it is actually a finite sum. The absolute convergence of the series on the left-hand side of (3.1) is obtained by the following argument. By Lemma 3.1, both $\{E_{mb}T_{na} f : m, n \in \mathbb{Z}\}$ and $\{E_{mb}T_{na} h : m, n \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R})$. Noting that

$$\langle f, E_{mb} T_{na}\, g\rangle = \mathrm{e}^{-2\pi\mathrm{i}mnab}\langle E_{-mb}\, T_{-na}\, f, g\rangle, \quad m, n \in \mathbb{Z},$$

and

$$\langle h, E_{mb} T_{na}\, \gamma\rangle = \mathrm{e}^{-2\pi\mathrm{i}mnab}\, \langle E_{-mb}\, T_{-na}\, h, \gamma\rangle, \quad m, n \in \mathbb{Z},$$

it follows that both $\{\langle f, E_{mb}\, T_{na}\, g\rangle\}_{m,n\in\mathbb{Z}}$ and $\{\langle h, E_{mb}\, T_{na}\, \gamma\rangle\}_{m,n\in\mathbb{Z}}$ are in $l^2(\mathbb{Z}^2)$. The series on the left-hand side of (3.1) thus converges absolutely using the Cauchy–Schwarz inequality, proving our claim. $\square$

Note that part (a) in the following theorem follows from [11, Corollary 2.4] under the weaker assumption that the corresponding system is complete in $L^2(S)$. However, as mentioned earlier, that result was obtained by more abstract methods of operator algebras and we prefer to give here a more direct proof of this result (which is needed to prove part (b) in any case) under the assumption that the system forms a frame. (See [3] for a similar proof in the case $S = \mathbb{R}$.)

Theorem 3.3. *Let $a, b > 0$, let S be an $a\,\mathbb{Z}$-shift invariant measurable set in $\mathbb{R}$ with nonzero measure, and suppose that $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is a frame for $L^2(S)$. Then,*

(a) $b\,|S_0| \le 1$, *where* $S_0 = S \cap [0, a)$.
(b) $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ *is a Riesz basis for* $L^2(S)$ *if and only if* $b\,|S_0| = 1$.

Proof. We denote by $\mathcal{S}$ the frame operator:

$$\mathcal{S} f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}\, T_{na}\, g\rangle\, E_{mb}\, T_{na}\, g, \qquad f \in L^2(S)$$

and $\gamma^\circ = \mathcal{S}^{-1} g$. Then

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle\, E_{mb}\, T_{na}\, \gamma^\circ, \qquad f \in L^2(S). \tag{3.2}$$

Define $G_n = \sum_{k\in\mathbb{Z}} \overline{g(\cdot - n/b - k\,a)}\, \gamma^\circ(\cdot - k\,a)$ for each $n \in \mathbb{Z}$. Suppose $f, h \in L^\infty(\mathbb{R})$ both vanish outside the set $S_0 \cap I$, where I is an interval of length $1/b$. Note that $T_{la}\, f$ and $T_{la}\, h$ both belong to $L^2(S)$ whenever $l \in \mathbb{Z}$. It follows from identity (3.2) and Lemma 3.2 that

$$\begin{aligned}
\langle f, h\rangle = \langle T_{la} f, T_{la} h\rangle &= \sum_{m,n\in\mathbb{Z}} \langle T_{la} f, E_{mb}\, T_{na}\, g\rangle\, \langle E_{mb}\, T_{na}\, \gamma^\circ, T_{la}\, h\rangle \\
&= \frac{1}{b} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} G_n(x) f\,(x - la - n/b)\, \overline{h\,(x - l\,a)} \mathrm{d}x \\
&= \frac{1}{b} \int_{\mathbb{R}} G_0(x) f\,(x - l\,a)\, \overline{h(x - l\,a)} \mathrm{d}x \\
&= \frac{1}{b} \int_{S_0\cap I} G_0(x + l\,a) f(x) \overline{h(x)} \mathrm{d}x.
\end{aligned}$$

Since f and g are arbitrary functions in $L^\infty(\mathbb{R})$ vanishing outside the set $S_0 \cap I$, it follows that $G_0(\cdot + l\,a) = b$ a.e. on $S_0 \cap I$ and thus also on S_0, since I is an arbitrary interval of length $1/b$. Hence, $G_0 = b$ a.e. on S and, since the functions $T_{na}\, \gamma^\circ$, $n \in \mathbb{Z}$, all belong to $L^2(S)$, G_0 vanishes outside of S. Hence, we conclude that $G_0 = b\, \chi_S$. This implies, in particular, that

$$\begin{aligned}
b\,|S_0| &= \int_{[0,a)} b\, \chi_S(x) \mathrm{d}x = \int_{[0,a)} G_0(x) \mathrm{d}x \\
&= \int_{[0,a)} \sum_{k\in\mathbb{Z}} \overline{g(x - k\,a)}\, \gamma^\circ(x - k\,a) \mathrm{d}x \\
&= \int_{\mathbb{R}} \overline{g(x)}\, \gamma^\circ(x) \mathrm{d}x = \langle \mathcal{S}^{-1} g, g\rangle = \langle \mathcal{S}^{-1/2}\, g, \mathcal{S}^{-1/2}\, g\rangle = \|\mathcal{S}^{-\frac{1}{2}}\, g\|^2.
\end{aligned}$$

It follows that

$$\|\mathcal{S}^{-\frac{1}{2}}g\|^2 = b|S_0|. \tag{3.3}$$

Since $\{E_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ is a frame for $L^2(S)$, it is well-known that the collection $\{\mathcal{S}^{-\frac{1}{2}} E_{mb} T_{na} g : m,n \in \mathbb{Z}\}$, which can also be written as $\{E_{mb} T_{na} \mathcal{S}^{-\frac{1}{2}} g : m,n \in \mathbb{Z}\}$ is a Parseval tight frame for $L^2(S)$. This implies that $\|\mathcal{S}^{-\frac{1}{2}}g\|^2 \leq 1$, which together with (3.3) shows that $b\,|S_0| \leq 1$ and proves (a). The collection $\{E_{mb} T_{na} g : m,n \in \mathbb{Z}\}$ forms a Riesz basis for $L^2(S)$ if and only if the Parseval tight frame $\{E_{mb} T_{na} \mathcal{S}^{-\frac{1}{2}} g : m,n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(S)$, which is equivalent to $\|\mathcal{S}^{-\frac{1}{2}}g\|^2 = 1$ or to $b\,|S_0| = 1$, using (3.3). This proves (b) and completes the proof. □

Remark. In the rational case, where $a\,b = \frac{p}{q}$ and $\gcd(p,q) = 1$, and under the hypotheses of the previous theorem, the condition $b\,|S_0| = 1$ to have a Riesz basis can also be written as

$$\sum_{k=0}^{p-1} \chi_S\left(\cdot + \frac{k}{b}\right) = q \quad \text{a.e. on } \mathbb{R},$$

as the proof of inequality (2.13) easily shows.

4. The irrational case

Our main goal, in this last section, is to show that, in the irrational case, if the condition $b\,|S \cap [0,a)| \leq 1$ holds, we can construct a measurable set $E \subset S$ whose $a\,\mathbb{Z}$-translates tile S and such that the Gabor system with window $g = \chi_E$ and parameters a, b forms a tight frame for $L^2(S)$. We will first need the following lemma.

Lemma 4.1. *Let a be an irrational number and suppose that E is a measurable subset of $\mathbb{R}$ which is both a $\mathbb{Z}$ and $\mathbb{Z}$-shift invariant. Then $E = \mathbb{R}$ or $\emptyset$ up to a set of zero measure.*

Proof. Since χ_E is 1-periodic, we can express it as a Fourier series

$$\chi_E(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}, \quad x \in \mathbb{R},$$

for some sequence $\{c_k\} \in \ell^2(\mathbb{Z})$ where the series converges locally in L^2. Since χ_E is also a-periodic, we have

$$\chi_E(x) = \chi_E(x+a) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k a} e^{2\pi i k x},$$

and the uniqueness of the Fourier coefficients implies that $c_k(1 - e^{2\pi i k a}) = 0$ for all integers k. Since a is irrational, this is equivalent to $c_k = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$, i.e. to $\chi_E = c_0$ a.e. on $\mathbb{R}$ from which our claim follows. □

Theorem 4.2. *Let a, $b > 0$ be such that $a\,b \notin \mathbb{Q}$. Let S be an $a\,\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ with nonzero measure and satisfying $b\,|S_0| \leq 1$, where $S_0 = S \cap [0,a)$. Then, there exists a measurable set E in $\mathbb{R}$ which is $a\,\mathbb{Z}$-congruent to S_0 and such that the collection $\{E_{mb}\, T_{na}\, \chi_E : m,n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.*

Proof. By the same argument as that of Theorem 2.12, we can assume that $b = 1, a > 1$ without loss of generality, and we only need to prove the existence of measurable subset E of $\mathbb{R}$ which is $a\,\mathbb{Z}$-congruent to S_0 and such that the collection $\left\{e^{2\pi i m x}\,\chi_E(x) : m \in \mathbb{Z}\right\}$ is a tight frame for $L^2(E)$. Let $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ and define a bijection $\sigma : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$\sigma(0) = 0, \quad \sigma(2k-1) = k, \quad \sigma(2k) = -k \quad \text{for}\, k > 0.$$

Let $g_0 = \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k)$. If $g_0 \le 1$ a.e. on $\mathbb{R}$, then $E = S_0$ is as desired. Otherwise, we proceed with the following inductive procedure to construct E, analogous to the construction given in Theorem 2.12 (and with similar notations). Let $H_0 = \{g_0 > 0\}$ and let T_0 be a measurable subset of S_0 such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_0}(\cdot + k) = \chi_{H_0}.$$

If the sets S_l, T_l, H_l and the function g_l have already been constructed for all indices l with $0 \le l \le j-1$, we define $S_j = S_{j-1} \setminus T_{j-1}$,

$$g_j = \sum_{k\in\mathbb{Z}} \chi_{S_j}(\cdot + a\,\sigma(j) + k),$$

and $H_j = \{g_j > 0,\ g_0 = g_1 = \cdots = g_{j-1} = 0\}$. We then choose a measurable set T_j contained in S_j and such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_j}(\cdot + a\,\sigma(j) + k) = \chi_{H_j}.$$

Define $E = \bigcup_{j\in\mathbb{Z}^+} \left(T_j - a\,\sigma(j)\right)$. Note that the sets H_j, $j \in \mathbb{Z}^+$, are mutually disjoint. We have

$$\sum_{k\in\mathbb{Z}} \chi_E(\cdot + k) = \sum_{j\in\mathbb{Z}^+} \sum_{k\in\mathbb{Z}} \chi_{T_j}(\cdot + a\,\sigma(j) + k) = \sum_{j\in\mathbb{Z}^+} \chi_{H_j} \le 1 \quad \text{a.e. on}\, \mathbb{R}.$$

So, by Lemma 2.10, the collection $\{E_{mb}\,\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(E)$. Since $T_j - a\,\sigma(j) \subset S_0 - a\,\sigma(j)$ for $j \in \mathbb{Z}^+$, and the sets $S_0 - a\,\sigma(j)$, $j \in \mathbb{Z}^+$, are mutually disjoint, so are the sets $T_j - a\,\sigma(j)$, $j \in \mathbb{Z}^+$. Hence, in order to prove that E is $a\,\mathbb{Z}$-congruent to S_0, we only need to prove that $\left|S_0 \setminus \left(\cup_{j\in\mathbb{Z}^+} T_j\right)\right| = 0$. Write $Q = S_0 \setminus \left(\cup_{j\in\mathbb{Z}^+} T_j\right)$. Since, for $j \ge 1$, $S_j = S_0 \setminus \left(\cup_{i=0}^{j-1} T_i\right)$, we have $Q \subset S_j$, for all $j \ge 0$, and thus

$$\begin{aligned}\left\{\sum_{k\in\mathbb{Z}} \chi_Q(\cdot + k) > 0\right\} &\subset \left\{\sum_{k\in\mathbb{Z}} \chi_{S_j}(\cdot + k) > 0\right\} = \left\{g_j(\cdot - a\,\sigma(j)) > 0\right\} \\ &\subset \bigcup_{m=0}^{\infty} (H_m + a\,\sigma(j)).\end{aligned}$$

It follows, using the disjointness the sets H_m, $m \ge 0$, that

$$\begin{aligned}\left\{\sum_{k\in\mathbb{Z}} \chi_Q(\cdot + k) > 0\right\} &\subset \bigcap_{j=0}^{\infty} \bigcup_{m=0}^{\infty} (H_m + a\,\sigma(j)) = \bigcap_{l\in\mathbb{Z}} \bigcup_{m=0}^{\infty} (H_m + a\,l) \\ &= \bigcap_{l\in\mathbb{Z}} \left(\bigcup_{m=0}^{\infty} \left\{\chi_{H_m}(\cdot - a\,l) = 1\right\}\right) = \bigcap_{l\in\mathbb{Z}} \left(\left\{\sum_{m\ge 0} \chi_{H_m}(\cdot - a\,l) = 1\right\}\right).\end{aligned}$$

Hence,

$$\left\{\sum_{k\in\mathbb{Z}} \chi_Q(\cdot + k) > 0\right\} \subset \bigcap_{l\in\mathbb{Z}} \left\{\sum_{m\geq 0}\sum_{k\in\mathbb{Z}} \chi_{T_m}(\cdot + a\,\sigma(m) - a\,l + k) = 1\right\} := \widetilde{Q}.$$

We will now prove that $|Q| = 0$ by contradiction. Suppose that $|Q| > 0$. Then,

$$\left|\bigcup_{m=0}^{\infty} (T_m + a\,\sigma(m))\right| = \sum_{m=0}^{\infty} |T_m + a\,\sigma(m)| = \sum_{m=0}^{\infty} |T_m| < |S_0| \leq 1 \tag{4.1}$$

due to the disjointness of the sets T_m, $m \geq 0$. It is obvious that

$$\widetilde{Q} \subset \left\{\sum_{m\geq 0}\sum_{k\in\mathbb{Z}} \chi_{T_m}(\cdot + a\,\sigma(m) + k) = 1\right\} = \left\{\sum_{k\in\mathbb{Z}} \chi_{\bigcup_{m=0}^{\infty} T_m - a\,\sigma(m)}(\cdot + k) = 1\right\}$$

since the sets $T_m - a\,\sigma(m)$, $m \geq 0$, are disjoint. Since $\widetilde{Q}$ is $\mathbb{Z}$-periodic and

$$|\widetilde{Q} \cap [0, 1)| = \int_{[0,1)} \chi_{\widetilde{Q}}(t)\mathrm{d}t \leq \int_{[0,1)} \sum_{k\in\mathbb{Z}} \chi_{\bigcup_{m=0}^{\infty} T_m - a\,\sigma(m)}(\cdot + k)\mathrm{d}t < 1,$$

using (4.1), it follows that $\widetilde{Q} \neq \mathbb{R}$ modulo a set of zero measure. However, $\tilde{Q}$ is both $a\,\mathbb{Z}$ and $\mathbb{Z}$-periodic and Lemma 4.1 shows that $|\tilde{Q}| = 0$. Therefore, we conclude that $|\{\sum_{k\in\mathbb{Z}} \chi_Q(\cdot + k) > 0\}| = 0$ and thus

$$|Q| = \int_{[0,1)} \sum_{k\in\mathbb{Z}} \chi_Q(t + k)\mathrm{d}t = 0,$$

which is a contradiction. The proof is completed. □

Corollary 4.3. *Let $a, b > 0$ be such that $a\,b \notin \mathbb{Q}$. Let S be an $a\,\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ and define the set $S_0 = S \cap [0, a)$. Then, the following are equivalent:*

(a) *There exists a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$.*
(b) *There exists a measurable set E in $\mathbb{R}$ which is $a\,\mathbb{Z}$-congruent to $S \cap [0, a)$ such that $\{E_{mb}T_{na}\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.*
(c) $b\,|S_0| \leq 1$.

Proof. As we mentioned earlier, the fact that (a) implies (c) is a result from [11]. The fact that (b) follows from (c) is the content of Theorem 4.2 and, clearly (b) implies (a). □

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